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Total Hamiltonian and Extended Hamiltonian for Constrained Hamilton Systems

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Abstract For constrained Hamiltonian systems, the motion equations are deduced from total Hamiltonian and extended Hamiltonian with Lagrangian multipliers depending on time t and canonical variables q^i and p_i. When the multipliers reduced to only depend on time t, the motion equations exactly agree with the old results. Under the same conditions (Lagrangian multipliers depend on time t and canonical variables q^i and p_i), the relation equations of coefficients in the generator of gauge transformation are deduced, but the equations have an additive term besides the well-known results. This additive term is from Lagrangian multipliers depending on canonical variables, and it might perform the gauge symmetries that needs to be discussed further.

Keywords Constrained Hamiltonian system · Total Hamiltonian · Extended Hamiltonian · Canonical Hamiltonian equations · Gauge symmetry

1 Introduction

For constrained Hamiltonian systems, the validity of Dirac's conjecture has been discussed more than half a century since Dirac brought it out [1, 2]. A system with a singular Lagrangian (a constrained Hamiltonian system) its canonical formalism is characterized by the presence of certain constraints about the canonical variables following the definition of generalized momentum. These constraints are called primary constraints, including possibly first-class and second-class. According to Dirac-Bergman method [3], the stationarity of all the primary constraints can produce secondary constraints (also including possibly first-class

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and second-class), or determine the corresponding Lagrangian multipliers, or only give some trivial equations. In Dirac's sense, all the secondary constraints should be introduced into the Hamiltonian, which is called extended Hamiltonian different from total Hamiltonian including only primary constraints. Based on the constraints, many problems were put forward. The relations between constraints and invariance were clarified by Bergman and co-workers [4, 5]. The interrelation between Lagrangian and Hamiltonian constraints were established by Kamimura [6]. The influence of singularity on the Lagrange equations and the formulation of Hamiltonian formalism were discussed carefully by Shanmugadahasan [7, 8]. The structure of Dirac bracket was investigated in detail by Sudarshan and Mukunda [9].

According to the kinds of the constraints, a constrained system can be taken as two kinds, one is the system only including first-class constraints, the other is that including both first-class and second-class ones. The former had been discussed in [10, 11] carefully. The later had been discussed in [12–14], too. In these discussions, Lagrangian multipliers were taken as constants, or time-dependent functions. In recent years, based on Lagrangian multipliers depending on time t and canonical variables q^i and p_i, some of characteristics of the constrained systems had been re-discussed [15–21], and master equations on the coefficients of the generator were given [22]. But considered Lagrangian multipliers as functions depending time t and canonical variable q^i and p_i, the motion equations of total Hamilton and extended Hamiltonian have not been done. In this paper, the equations of motion will be deduced, and the equations of coefficients of the generator will be re-deduced, based on total Hamiltonian and extended Hamiltonian. The motion equations can be reduced to the same form as known, but the relation equations of coefficients have an additive term, which is from the Lagrangian multipliers depending on canonical variables, and might perform in gauge symmetry. This might need to be discuss further.

2 In Total Hamiltonian Formalism

For simplicity, let us consider a constrained mechanical system with a finite number of degrees of freedom its motion is described by a Lagrangian $L(t; q^i, \dot{q}^i)$. Variables q^i $(i = 1, 2, \dots, n)$ are called generalized coordinates and $\dot{q}^i$ are called generalized velocities. And the Lagrangian L is dependent on time t explicitly. According to the definition of canonical momentum, $p_i = \frac{\partial L}{\partial \dot{q}^i}$, we can give primary constraints as

$$\phi_\rho^0 = p_\rho - g_\rho(q^i, p_a) \quad (a = 1, 2, \dots, R;\ \rho = 1, 2, \dots, n - R) \tag{1}$$

where ϕ_ρ^0 do depend on the generalized coordinates q^i and generalized momenta p_i (including p_a and p_ρ), but not on the generalized velocities $\dot{q}^i$, which are called primary constraints. In Dirac's sense, the total Hamiltonian can be written as

$$H_T = H_C + \lambda^\rho(t; q^i, p_i)\phi_\rho^0 \tag{2}$$

where H_C denotes canonical Hamiltonian, $H_C = p_i\dot{q}^i - L(t; q^i, p_i)$; ϕ_ρ^i are primary constraints; and $\lambda^\rho(t; q^i, p_i)$ denote Lagrangian multipliers introduced corresponding to the constraints, respectively. The motion equations of total Hamiltonian can be expressed as

$$\dot{q}^i = \{q^i, H_T\} = \{q^i, H_C\} + \{q^i, \lambda^\rho(t; q^i, p_i)\}\phi_\rho^0 + \lambda^\rho(t; q^i, p_i)\{q^i, \phi_\rho^0\} \tag{3}$$

$$\dot{p}_i = \{p_i, H_T\} = \{p_i, H_C\} + \{p_i, \lambda^\rho(t; q^i, p_i)\}\phi_\rho^0 + \lambda^\rho(t; q^i, p_i)\{p_i, \phi_\rho^0\} \tag{4}$$

here $\{\cdot,\cdot\}$ denotes the Poisson bracket. Any time-dependent and canonical variable-dependent function $F(t;q^i,p_i)$, whose evolution of time can be given as

$$\dot{F}=\{F,H_T\}=\{F,H_C\}+\{F,\lambda^\rho(t;q^i,p_i)\}\phi_\rho^0+\lambda^\rho(t;q^i,p_i)\{F,\phi_\rho^0\} \tag{5}$$

If we assumed that in Dirac's conjecture the first-class constraints are limited in primary constraints, the generator of gauge transformation would be performed as

$$G=\varepsilon^l(t)\phi_l^0 \tag{6}$$

where $\varepsilon^l(t)$ denote arbitrary coefficients in the generator of gauge transformation, and which are functions depending only on time t, corresponding to the primary first-class constraints ϕ_l^0, respectively. According to the stationarity of the generator $\dot{G}=\frac{\partial G}{\partial t}+\{G,H_T\}=0$, the relations between coefficients and Lagrangian multipliers can be obtained as

$$\dot{\varepsilon}^l(t)\phi_l^0+\varepsilon^l(t)\{\phi_l^0,H_C\}+\varepsilon^l(t)\{\phi_l^0,\lambda^\rho(t;q^i,p_i)\}\phi_\rho^0=0 \tag{7}$$

here the definition of first-class constraints and the stationarity of the constraints have been used. If the Lagrangian multipliers $\lambda^\rho(t;q^i,p_i)$ are simplified as $\lambda^\rho(t)$ which are depending only on time t, the (7) would be simplified as the results given by Galvão and Boechat [23]

$$\frac{d\varepsilon^l(t)}{dt}\phi_l^0+\varepsilon^l(t)\{\phi_l^0,H_C\}=0 \tag{8}$$

If the constrained system does not have secondary first-class constraints, in other words, all secondary constraints are second-class constraints or can be expressed by the linear combination of the primary first-class, (7) could be rewritten as

$$\frac{d\varepsilon^l(t)}{dt}+\varepsilon^n(t)\alpha_{nl}=0 \tag{9}$$

where α_{nl} are defined as $\{\phi_l^0,H_C\}=\alpha_{nl}\phi_l^0$. If both of the coefficients $\varepsilon^l(t;q^i,p_i)$ and the Lagrangian multipliers $\lambda^\rho(t;q^i,p_i)$ are depending on time t and canonical variables q^i and p_i, how did the master equations in [20–22] express need discussing further [24].

In mathematical sense, the motions of a constrained system must be limited on the hypersurface of constraints, but in physical sense, they do not, once other external forces perform. Therefore, in mathematical sense (or in physical sense without external force performing), the total Hamiltonian equations can be simplified as the following

$$\dot{q}^i=\{q^i,H_T\}=\{q^i,H_C\}+\lambda^\rho(t)\{q^i,\phi_\rho^0\} \tag{10}$$

$$\dot{p}_i=\{p_i,H_T\}=\{p_i,H_C\}+\lambda^\rho(t)\{p_i,\phi_\rho^0\} \tag{11}$$

Both of the above equations are equivalent to the canonical Hamiltonian equations when Lagrangian multipliers $\lambda^l(t)$ depended only on time t very well. And for function $F(t;q^i,p_i)$, its evolution can be given as

$$\dot{F}=\{F,H_T\}=\{F,H_C\}+\lambda^\rho(t)\{F,\phi_\rho^0\} \tag{12}$$

exactly agreeing with the result when Lagrangian multipliers are depending only on time t.

Under most conditions, the dynamical system is developing continuously, i.e., there do not appear that an external force or source is suddenly introduced. When Lagrangian multipliers $\lambda^{\rho}(t; q^i, p_i)$ depend on time t and canonical variables q^i and p_i, whether the motion equations of extended Hamiltonian can agree with (10), (11) and (12) will be discussed in what follows.

3 In Extended Hamiltonian Formalism

For simplicity, we still consider the above system with Lagrangian $L(t; q^i, \dot{q}^i)$. For this constrained system, there are primary constraints ϕ_l^0 and total Hamiltonian H_T, which have been discussed in the above section. Following Dirac-Bergman approach, secondary constraints can be derived from the stationarity of primary constraints. The stationarity can be written as

$$\frac{d\phi_l^0}{dt} = \frac{\partial \phi_l^0}{\partial t} + \{\phi_l^0, H_T\} = 0 \tag{13}$$

For the primary constraints do not depend on time t explicitly, (13) can be rewritten as

$$\frac{d\phi_l^0}{dt} = \{\phi_l^0, H_T\} = \{\phi_l^0, H_C\} + \lambda^m(t; q^i, p_i)\{\phi_l^0, \phi_m^0\} \approx 0 \tag{14}$$

where "$\approx$" [25–27] different from "$=$" [28, 29] denotes (14) limited on hyper-surface of primary constraints. Therefore, the above equations can be rewritten as

$$\{\phi_l^0, H_C\} + \lambda^m(t; q^i, p_i)\{\phi_l^0, \phi_m^0\} \approx 0 \tag{15}$$

If ϕ_m^0 are second-class constraints, we can get $|\{\phi_l^0, \phi_m^0\}| \neq 0$. From (15), the corresponding Lagrangian multipliers can be solved as

$$\lambda^m(t; q^i, p_i) = -\{\phi_l^0, H_C\}\{\phi_l^0, \phi_m^0\}^{-1} \tag{16}$$

Otherwise, (14) would be trivial identities, or would produce new constraints $\phi_p^1 = \{\phi_l^0, H_C\}$. This indicates that all secondary constraints naturally are generated by the commutation relations between known first-class constraints and canonical Hamiltonian, not from total Hamiltonian [30, 31]. The second-hand constraints are from the commutation relations between the primary first-class constraints and canonical Hamiltonian, the third-hand constraints are from the commutation relations between the second-hand first-class constraints and canonical Hamiltonian, and so on. Until the constraints do not give rise to other new constraints, this process would stop. In the last step, we will get identities or linear combinations of known constraints.

In Dirac's sense, the motion of a constrained system should be described by extended Hamiltonian which is introduced all constraints (including primary and secondary ones). Therefore, (3) and (4) should be replaced by

$$\begin{aligned}\dot{q}^i = \{q^i, H_E\} = \{q^i, H_C\} &+ \lambda^s(t; q^i, p_i)\{q^i, \chi_s\} + \{q^i, \lambda^s(t; q^i, p_i)\}\chi_s \\ &+ \{\theta_l, H_C\}\{\theta_l, \theta_m\}^{-1}\{q^i, \theta_m\}\end{aligned} \tag{17}$$

$$\begin{aligned}\dot{p}_i = \{p_i, H_E\} = \{p_i, H_C\} &+ \lambda^s(t; q^i, p_i)\{p_i, \chi_s\} + \{p_i, \lambda^s(t; q^i, p_i)\}\chi_s \\ &+ \{\theta_l, H_C\}\{\theta_l, \theta_m\}^{-1}\{p_i, \theta_m\}\end{aligned} \tag{18}$$

where χ_s denote all first-class constraints including primary and secondary ones; θ_m denote all second-class constraints including primary and secondary ones, too; H_E denotes extended Hamiltonian $H_E = H_C + \lambda^s(t; q^i, p_i)\chi_s + \lambda^n(t; q^i, p_i)\theta_m$. In above equations, all of the Lagrangian multipliers $\lambda^m(t; q^i, p_i)$ corresponding to second-class constraints are replaced by their solutions.

Under the same conditions, the evolution equation function $F(t; q^i, p_i)$ is re-written as

$$\dot{F} = \{F, H_E\} = \{F, H_C\} + \lambda^s(t; q^i, p_i)\{F, \chi_s\} + \{F, \lambda^s(t; q^i, p_i)\}\chi_s + \{\theta_l, H_C\}\{\theta_l, \theta_m\}^{-1}\{F, \theta_m\} \tag{19}$$

Selecting the assumption in the above section (Lagrangian multipliers $\lambda^s(t)$ only depend on time t and all motions are limited on hyper-surface defined by all constraints), we can simplify (17), (18) and (19) as

$$\dot{q}^i = \{q^i, H_E\} = \{q^i, H_C\} + \lambda^s(t)\{q^i, \chi_s\} + \{\theta_l, H_C\}\{\theta_l, \theta_m\}^{-1}\{q^i, \theta_m\} \tag{20}$$

$$\dot{p}_i = \{p_i, H_E\} = \{p_i, H_C\} + \lambda^s(t)\{p_i, \chi_s\} + \{\theta_l, H_C\}\{\theta_l, \theta_m\}^{-1}\{p_i, \theta_m\} \tag{21}$$

$$\dot{F} = \{F, H_E\} = \{F, H_C\} + \lambda^s(t)\{F, \chi_s\} + \{\theta_l, H_C\}\{\theta_l, \theta_m\}^{-1}\{F, \theta_m\} \tag{22}$$

The three equations have exactly the same form of the results given by Li group [32]. If we do not limited the motions on the hyper-surface, the above motion equations would be added another term $\{\cdot, \lambda^s(t; q^i, p_i)\}\chi_s$. Furthermore, they are very difficultly simplified as the results in Ref. [32]. Whereas we will give some simple discussion about this problem. We first simplify the additional term as

$$\{q^i, \lambda^s(t; q^i, p_i)\}\chi_s = \frac{\partial\lambda^s(t; q^i, p_i)}{\partial p_i}\chi_s \tag{23}$$

$$\{p_i, \lambda^s(t; q^i, p_i)\}\chi_s = -\frac{\partial\lambda^s(t; q^i, p_i)}{\partial q^i}\chi_s \tag{24}$$

$$\{F, \lambda^s(t; q^i, p_i)\}\chi_s = \left[\frac{\partial F}{\partial q^i}\frac{\partial\lambda^s(t; q^i, p_i)}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial\lambda^s(t; q^i, p_i)}{\partial q^i}\right]\chi_s \tag{25}$$

According to the Lagrangian multipliers $\lambda^s(t; q^i, p_i)$ are arbitrary functions, which are depending on time t and canonical variables q^i and p_i, we redefine three new series of Lagrangian multipliers $\lambda^{s\prime}(t; q^i, p_i)$, $\lambda^{s\prime\prime}(t; q^i, p_i)$ and $\lambda^{s\prime\prime\prime}(t; q^i, p_i)$ as the following

$$\lambda^{s\prime}(t; q^i, p_i) = \lambda^s(t; q^i, p_i) + \frac{\partial\lambda^s(t; q^i, p_i)}{\partial p_i} \tag{26}$$

$$\lambda^{s\prime\prime}(t; q^i, p_i) = \lambda^s(t; q^i, p_i) - \frac{\partial\lambda^s(t; q^i, p_i)}{\partial q^i} \tag{27}$$

$$\lambda^{s\prime\prime\prime}(t; q^i, p_i) = \lambda^s(t; q^i, p_i) + \left[\frac{\partial F}{\partial q^i}\frac{\partial\lambda^s(t; q^i, p_i)}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial\lambda^s(t; q^i, p_i)}{\partial q^i}\right] \tag{28}$$

Therefore, (17), (18) and (19) can be rewritten as

$$\dot{q}^i = \{q^i, H_E\} = \{q^i, H_C\} + \lambda^{s'}(t; q^i, p_i)\{q^i, \chi_s\} + \{\theta_l, H_C\}\{\theta_l, \theta_m\}^{-1}\{q^i, \theta_m\} \tag{29}$$

$$\dot{p}_i = \{p_i, H_E\} = \{p_i, H_C\} + \lambda^{s''}(t; q^i, p_i)\{p_i, \chi_s\} + \{\theta_l, H_C\}\{\theta_l, \theta_m\}^{-1}\{p_i, \theta_m\} \tag{30}$$

$$\dot{F} = \{F, H_E\} = \{F, H_C\} + \lambda^{s'''}(t; q^i, p_i)\{F, \chi_s\} + \{\theta_l, H_C\}\{\theta_l, \theta_m\}^{-1}\{F, \theta_m\} \tag{31}$$

If there are a series of Lagrangian multipliers κ^s, which can denote all $\lambda^{s'}(t; q^i, p_i)$, $\lambda^{s''}(t; q^i, p_i)$ and $\lambda^{s'''}(t; q^i, p_i)$, the above equations would exactly agree with the results in [32–34].

In Dirac's sense, all first-class constraints are the generator of gauge transformation. The generator should be expressed as

$$G = \omega^s(t)\chi_s \tag{32}$$

where $\omega^s(t)$ denote coefficients of the generator of the gauge transformation corresponding to first-class constraints, respectively; χ_s denote all first-class constraints (including the primary ones and the secondary ones). According to the stationarity of generator $\frac{dG}{dt} = \frac{\partial G}{\partial t} + \{G, H_E\} = 0$, the relations between the coefficients $\omega^s(t)$ and the Lagrangian multiplier $\lambda^t(t; q^s, p_i)$ can be given as

$$\dot{\omega}^s(t)\chi_s + \omega^s(t)\{\chi_s, H_C\} + \omega^s(t)\{\chi_s, \lambda^t(t; q^i, p_i\}\chi_t + \omega^s(t)\lambda^t(t; q^i, p_i)\{\chi_s, \chi_t\} = 0 \tag{33}$$

In the definition of first-class constraint, the third term in (33) is vanished. And by defining $\{\chi_s, H_C\} = \alpha_{s\mu}\chi^\mu$ and $\{\chi_s, \lambda^t(t; q^i, p_i\} = \beta_{st}\chi^t$, and limiting the evolution of generator on hyper-surface, we can simplify the above equations as the following

$$\frac{d\omega^s(t)}{dt} + \omega^\mu(t)\alpha_{\mu s} + \omega^t(t)\beta_{ts} = 0 \tag{34}$$

This equations do not agree with the results in [35–38]. There is an additive term $\omega^t(t)\beta_{ts}$.

4 Conclusions

In this paper, when Lagrangian multipliers are functions depending on time t and canonical variables q^i and p_i, the motion equations of total Hamiltonian and extended Hamiltonian have been deduced. The relations of between the total Hamiltonian and extended Hamiltonian equations and the canonical Hamiltonian ones are discussed. The situation of the Lagrangian multipliers depending on time t and canonical variables q^i and p_i is more general than that of the Lagrangian multipliers depending only on time. And the relation equations of coefficients of generator have been deduced, too. There is an additive term in the equations, this result might lead to affect the gauge transformation, even bring out the alteration of the corresponding conserved charge. Once this assumption is proved right, the validity of Dirac's conjecture would be invalid. In other words, some secondary first-class constraints may not be generators of gauge transformation in a constrained Hamiltonian system with a singular Lagrangian. This assumption needs to be discussed further. Therefore, the total Hamiltonian including all primary constraints (first-class and second-class constraints) is more general than the extended Hamiltonian including all constraints (primary first-class and second-class constraints, and secondary first-class and second-class ones), at least before Dirac's conjecture is proved valid.

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References

1. Dirac, P.A.M.: Can. J. Math. **2**, 129 (1950)
2. Dirac, P.A.M.: Lecture on Quantum Mechanics. Yeshiva University, New York (1964)
3. Gitman, D.M., Tyutin, I.V.: Quantization of Field with Constraints. Springer, Berlin (1990)
4. Anderson, J.L., Bergmann, P.G.: Phys. Rev. **83**, 1018 (1951)
5. Bergmann, P.G., Goldberg, J.: Phys. Rev. **98**, 531 (1955)
6. Kamimura, K.: Nuovo Cimento B **68**, 33 (1962)
7. Shanmugadahasan, S.: Proc. Camb. Phil. Soc. **59**, 743 (1963)
8. Shanmugadahasan, S.: J. Math. Phys. **14**, 677 (1973)
9. Sudarshan, E., Mukunda, N.: Classical Dynamics, A Modern Perspective. Wiley, New York (1974)
10. Henneaux, M., Teitelboim, C.: Quantization of Gauge System. Princeton University Press, Princeton (1992)
11. Cabo, A.: J. Phys. A: Gen. **19**, 629 (1986)
12. Costa, M.E.V., Girotti, H.O., Simoes, T.J.M.: Phys. Rev. D **32**, 405 (1985)
13. Li, A.M., Jiang, J.H., Li, Z.P.: Acta Phys. Sin. **51**, 945 (2002)
14. Li, Z.P., Li, A.M., Jiang, J.H., Wang, Y.L.: Comm. Theor. Phys. **43**, 1115 (2005)
15. Shirzad, A.: J. Phys. A: Gen. **31**, 2747 (1998)
16. Banerjee, R., Rothe, H.J., Rothe, K.D.: Phys. Lett. B **463**, 248 (1999)
17. Banerjee, R., Rothe, H.J., Rothe, K.D.: J. Phys. A: Gen. **33**, 2059 (2000)
18. Rothe, H.J., Rothe, K.D.: J. Phys. A: Gen. **36**, 1671 (2003); hep-th/0302210
19. Rothe, H.J., Rothe, K.D.: Ann. Phys. **313**, 479 (2004)
20. Rothe, H.J.: Phys. Lett. B **539**, 296 (2002)
21. Rothe, H.J.: Phys. Lett. B **569**, 90 (2003)
22. Banerjee, R., Rothe, H.J., Rothe, K.D.: Phys. Lett. B **479**, 429 (2000)
23. Galvão, C.A.P., Boechat, J.B.T.: J. Math. Phys. **30**, 1122 (1990)
24. Wang, Y.L., Li, Z.P.: Rediscussion of Gauge Symmetry (in preparation)
25. Qi, Z.: Int. J. Theor. Phys. **29**, 1309 (1990)
26. Li, Z.P., Li, X.: Int. J. Theor. Phys. **30**, 225 (1991)
27. Li, Z.P., Li, X.: Phys. Rev. E **52**, 876 (1994)
28. Cawley, R.: Phys. Rev. Lett. **42**, 413 (1979)
29. Cawley, R.: Phys. Rev. D **21**, 2988 (1980)
30. Li, Z.P.: Chin. Phys. Lett. **10**, 68 (1993)
31. Li, Z.P.: Europhys. Lett. **21**, 141 (1993)
32. Li, Z.L., Jiang, J.H.: Symmetries in Constrained Canonical Systems. Science Press, Beijing (2002)
33. Li, Z.P.: Classical and Quantal Dynamics of Constrained Systems and Their Symmetry Properties. Beijing Polytechnic University Press, Beijing (1993) (in Chinese)
34. Li, Z.P.: Constrained Hamiltonian Systems and Their Symmetry Properties. Beijing Polytechnic University Press, Beijing (1999) (in Chinese)
35. Li, Z.P.: J. Phys. A: Gen. **24**, 426 (1991)
36. Li, Z.P.: High Energy Phys. Nucl. Phys. **17**, 693 (1994) (in Chinese)
37. Wang, A.M., Ruan, T.N.: Phys. Rev. Lett. **73**, 2011 (1994)
38. Wang, A.M., Ruan, T.N.: Phys. Rev. A **54**, 57 (1996)